

Phase Coherence in Chaotic Oscillatory Media

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Abstract

Collective oscillations of lattices of locally-coupled chaotic Rössler oscillators are studied with regard to the dynamical scaling of their phase interfaces. Using analogies with the complex Ginzburg-Landau and the Kardar-Parisi-Zhang equations, we argue that phase coherence should be lost in the infinite-size limit. Our numerical results, however, indicate possible discrepancies with a Langevin-like description using an effective white-noise term.

Spatially-extended, extensively-chaotic dynamical systems with local interactions generically exhibit some collective coherence emerging out of strong local chaos which seems to persist even in the infinite-size limit. This long-range order in far-from-equilibrium, deterministic systems often takes the form of a simple, effectively low-dimensional, temporal evolution of spatially-averaged quantities. Usually referred to as non-trivial collective behavior, this phenomenon has been studied mostly in discrete-time lattice systems such as coupled map lattices and cellular automata [1]. In a recent paper, though, a continuous-time model was investigated in this context [2]. It concluded from numerical experiments that two-dimensional lattices of diffusively-coupled chaotic Rössler systems may show collective oscillations, or long-range rotating order (LRRO). In this Paper, we look at this result in a new light, by making use of the properties of the Kardar-Parisi-Zhang equation (KPZE) [3], a universal model for fluctuating interfaces.

1 Motivation

A lattice of Rössler systems coupled to their nearest neighbors by diffusion can be schematically written:

$$C_{t+\tau}^i = (1 - \varepsilon)\mathcal{F}_\tau(C_t^i) + \frac{\varepsilon}{\mathcal{N}} \sum_{j \in \mathcal{V}_i} \mathcal{F}_\tau(C_t^j), \quad (1)$$

where C^i is the three-component vector sitting at site i , ε is the coupling strength, τ is the interval between coupling times, \mathcal{N} is the number of nearest-neighbors, \mathcal{V}_i is the neighborhood of site i , and $\mathcal{F}_\tau(C_t^i)$ represents the state of C_t^i after evolution under the Rössler flow \mathcal{R} during a time τ . In other words, dropping the subscript i :

$$\mathcal{F}^\tau(C^t) = C^t + \int_t^{t+\tau} dt' \dot{C}. \quad (2)$$

The Rössler model possesses remarkable properties. It is usually written as a three-variable, first-order, ordinary differential system:

$$\dot{C} = \begin{pmatrix} \dot{c}_1 \\ \dot{c}_2 \\ \dot{c}_3 \end{pmatrix} = \begin{pmatrix} -c_2 - c_3 \\ c_1 + ac_1 \\ b + c_1c_3 - cc_3 \end{pmatrix} \equiv \mathcal{R}(C) \quad (3)$$

where a , b , and c are real parameters. Increasing c while keeping a and b fixed (for example $a = b = 0.2$ as in [2]), system (3) undergoes a Hopf bifurcation followed by a cascade of subharmonic bifurcations eventually leading to chaos. For these parameter values, the chaotic attractor is characterized by c_3 peaks of irregular amplitude but almost perfectly-defined frequency (Fig. 1a). Similarly, trajectories in the (c_1, c_2) plane are cycles of irregular amplitude but with a well-defined period (Fig. 1b). This allows the definition of “phase” and “amplitude” variables, either simply by using the (c_1, c_2) plane with an origin set in the middle of the attractor (Fig. 1b), or by more sophisticated methods. One can thus speak of a “chaotic oscillator” [5]. Picturing the Rössler system as an oscillator translates the problem of the LRRO observed in [2] into a phase coherence, or synchronization, problem for a chaotic oscillatory medium¹.

In [2], the chaotic oscillatory behavior of lattices of Rössler systems of the type (1-3) was also used to draw an analogy with the complex Ginzburg-Landau

¹ Note that the remarkable phase coherence of the Rössler model was recently studied and quantified in [5], where the possibility of (exact) synchronization of these systems was evidenced.

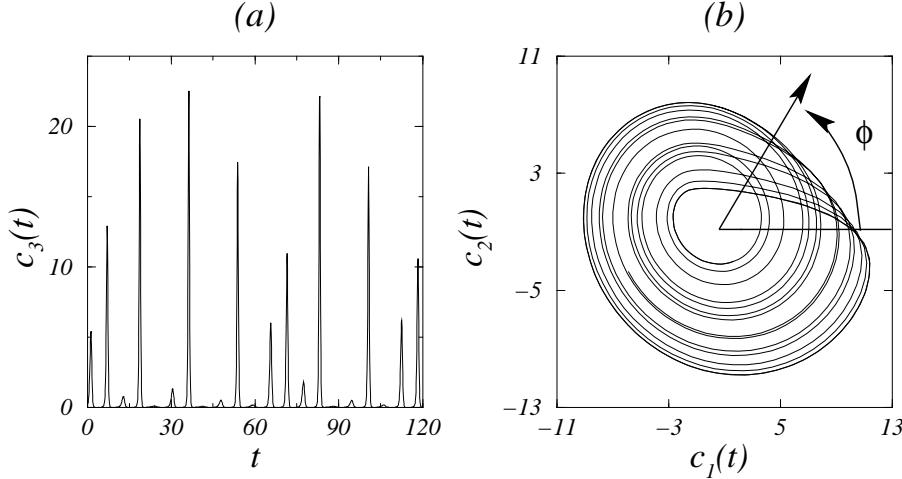


Fig. 1. Chaotic dynamics of the Rössler system (3) with $a = b = 0.2$ and $c = 5.7$. (a): time series of c_3 . (b): attractor in the (c_1, c_2) plane and definition of phase.

equation (CGLE), the generic nonlinear partial differential equation describing an oscillatory medium near a Hopf bifurcation [6]. The CGLE reads:

$$\partial_t A = A + (1 + i\alpha)\nabla^2 A - (1 + i\beta)|A|^2 A \quad (4)$$

where A is a complex field, α and β are real parameters. It was argued that the LRRO observed can be well accounted for by a CGLE with parameters corresponding to a regime where homogeneous oscillations are (linearly) stable, with some residual “effective” noise. In that analogy, the complex plane roughly corresponds to the (c_1, c_2) plane of the Rössler variables. On general grounds, one expects the soft phase modes of the noisy stable CGLE to be described at large scales by the Kardar-Parisi-Zhang equation (KPZE)², a stochastic model for the kinetic roughening of fluctuating interfaces [3] which reads:

$$\partial_t h = \nu \nabla^2 h + \frac{\lambda}{2} (\nabla h)^2 + \eta(\mathbf{x}, t) \quad (5)$$

where h is a real field, ν and λ are real parameters, and $\eta(\mathbf{x}, t)$ is an uncorrelated white noise with zero mean and correlators:

$$\langle \eta(\mathbf{x}, t) \eta(\mathbf{x}', t') \rangle = 2D \delta(t - t') \delta(\mathbf{x} - \mathbf{x}'). \quad (6)$$

In (5), h is the height of an interface and takes values on the entire real axis. In the present context, h represents the angular (or phase) argument of the complex field A (or the phase ϕ of the Rössler oscillators) followed

² At least when no zeroes of the complex field A are present and when the above-mentioned effective noise is delta-correlated.

by continuity in space and time from some arbitrary initial value. In this representation, the phase coherence problem described above can be restated in terms of the roughness of the phase interface: if the interface is rough (its mean square width diverges in the infinite-size infinite-time limit), then no phase coherence exists.

The KPZE has gained considerable importance because many “microscopic” models share its non-trivial scaling properties, and also because some analytical results have been obtained [3]. Among those, one is crucial here: interfaces governed by the KPZE are always rough for space dimensions $d \leq 2$. Assuming the validity of the KPZE to describe the large-scale properties of lattices of coupled Rössler oscillators, this result is at odds with the conclusion reached in [2]. Thus, either the numerical results of [2] were too limited to reveal the loss of phase coherence in the infinite-size, infinite-time limit, or the KPZE is not the correct stochastic equation. In the latter case, the discrepancy probably lies in the properties of the “effective noise”³.

As a matter of fact, there is no *a priori* reason for the KPZE to be the relevant large-scale description of the phase dynamics of coupled Rössler oscillator lattices. But the “universality class” of the KPZE has been shown to be remarkably large. In particular, recent findings show it to include the phase interface dynamics generated by the CGLE in its so-called “phase turbulence” regime [9]. In this spatiotemporally chaotic regime, there are no zeroes of the complex field A , and a fluctuating continuous phase interface can always be defined⁴. To some extent, this case might appear very similar to the chaotic oscillatory media formed by lattices of Rössler systems; thus, one would expect the KPZE to be relevant. On the other hand, as recalled above, chaos in the Rössler system possesses some very specific features that might possibly produce an “effective noise” with peculiar properties.

Given the space-time evolution of some interface, there exists a rather well-established procedure to determine whether the KPZE is a relevant large-scale description [7]. In the following, we briefly recall this procedure and follow it to investigate the coherent oscillations in lattices of coupled Rössler systems from the angle of the phase interface dynamics they produce.

³ The presence of the diffusive term is guaranteed by the coupling function in (1), and that of the quadratic nonlinear term is insured by the relevance of the CGLE demonstrated in [2]. Indeed, for the CGLE, the $(\nabla h)^2$ term represents the quadratic dependence of frequency on wavenumber implied by the nonlinear term $|A|^2 A$.

⁴ In fact, it was also concluded in [9] that phase turbulence always eventually breaks down, leading to the occurrence of zeroes of A . But there exist large portions of the parameter plane (α, β) in which statistically-steady phase turbulence can be observed on very large scales, so that, in practice, breakdown is never observed.

2 Experimental procedure

We have performed numerical experiments of system (1-3) and studied the scale-invariance properties of the interface constructed from the “phase” of each of the Rössler oscillators. The calculations were done in one space dimension, mostly for numerical convenience — if phase coherence is to be broken, it should be easier to observe for $d = 1$ — but also because many exact results are known in this case for the KPZE. Scale-invariance relies on the scaling assumption $h(\ell\mathbf{x}, \ell^z t) = \ell^\zeta h(\mathbf{x}, t)$, where ℓ is a similarity factor, ζ and z are (respectively) the roughness and dynamical exponents. For interfaces governed by the KPZE, exponents are exactly known for $d = 1$: $z = \frac{3}{2}$ and $\zeta = \frac{1}{2}$. All scaling laws given in the following are for these values.

In practice, one usually considers global quantities such as the mean square width of the interface:

$$w^2(t) = \left\langle (h(\mathbf{x}, t) - \langle h \rangle_{\mathbf{x}})^2 \right\rangle_{\mathbf{x}} , \quad (7)$$

where $\langle \dots \rangle_{\mathbf{x}}$ denotes space average. For a system of finite-size L , the width of an initially flat interface grows and saturates to a size-dependent mean value:

$$w^2(L, t \rightarrow \infty) = \frac{D}{24\nu} L . \quad (8)$$

For an infinitely large system, w^2 grows indefinitely:

$$w^2(L \rightarrow \infty, t) \simeq 0.4 \times \left(\frac{D^2}{4\nu^2} \lambda t \right)^{2/3} . \quad (9)$$

For the KPZE, this growth phase actually takes place only beyond certain crossover scales [8]:

$$L_c \simeq \frac{152}{g} \quad \text{and} \quad t_c \simeq \frac{252}{\nu g^2} \quad \text{with} \quad g = \frac{\lambda^2 D}{\nu^3} , \quad (10)$$

before which another scaling is observed, because the nonlinearities are not yet effective. The “linear” growth phase is characterized by a growth exponent $\beta = (2 - d)/4 = 1/4$ for $d = 1$. One expects:

$$w^2(L \rightarrow \infty, t) = \frac{D}{\sqrt{2\pi\nu}} t^{1/2} . \quad (11)$$

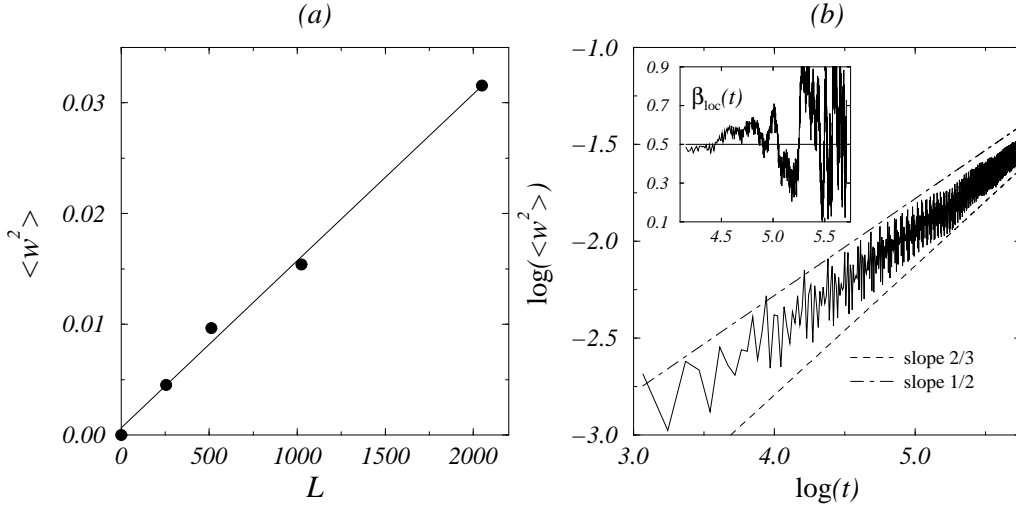


Fig. 2. Chain of diffusively coupled Rössler systems. (a): the ensemble-averaged, saturated, square mean width scales with system size L , with a slope $D/24\nu \simeq 1.5 \times 10^{-5}$. (b): in the growth regime, a system of 2^{18} oscillators seems to remain in the linear regime (single run). Lines of slope $1/2$ (linear regime) and $2/3$ (KPZE nonlinear regime) are shown. The insert shows the time evolution of the “local growth exponent” $\beta_{\text{loc}}(t)$ calculated over a time-window $\Delta t = 15000$. We cannot rule out the beginning of a crossover from the value $1/2$ to some larger value, but the data is too noisy to conclude.

Relations (8), (9), and (11) allow one to check dynamical scaling and to estimate whether the measured exponents are consistent with those of the KPZE. In addition, the measure of the numerical prefactors of the scaling laws can lead to a determination of the effective parameters ν , D , and λ of the corresponding KPZE and of the crossover scales L_c and t_c , provided that λ is determined independently. This is usually achieved by measuring the changes in the velocity of the interface $v = d\langle\phi\rangle_x/dt$ when it is submitted to a tilt $q = 2\pi n/L$, where n is an integer “winding number”, using the relation [7]:

$$\lambda = \left. \frac{d^2 v}{dq^2} \right|_{q=0}. \quad (12)$$

3 Chain of Rössler systems with purely diffusive coupling

We first consider a chain of Rössler systems with periodic boundary conditions, as defined by (1-3). The coupling strength and the coupling interval are set to $\varepsilon = 1/6$ and $\tau = 1.8$, values which insure the quasi-continuity of the medium and thus of the phase interface. The parameters of the Rössler systems themselves have the same values as in [2]: $a = b = 0.2$ and $c = 5.7$. Starting

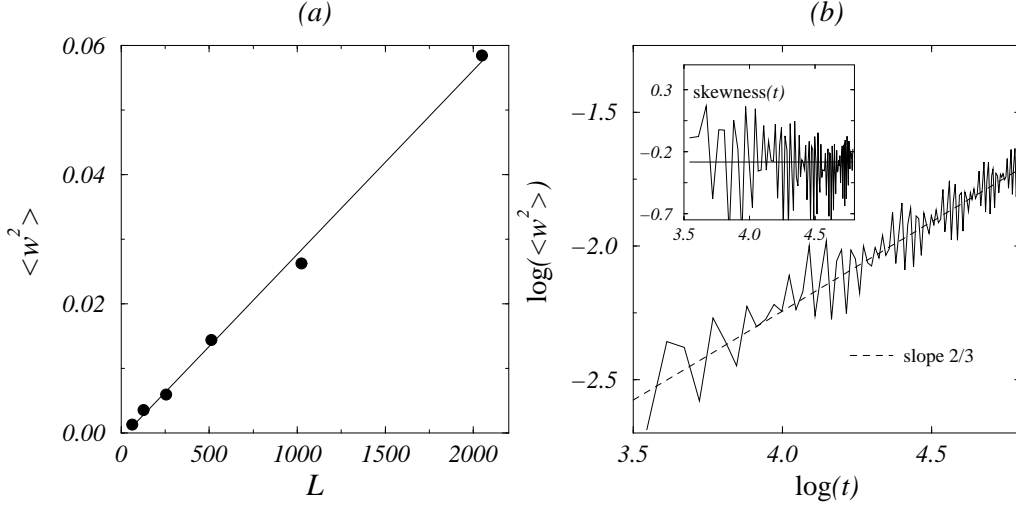


Fig. 3. Chain of Rössler systems with diffusive-dispersive coupling. (a): the ensemble-averaged, saturated, square mean width scales with system size L , with a slope $D/24\nu \simeq 2.85 \times 10^{-5}$. (b): in the growth regime, a system of 2^{18} oscillators quickly reaches the KPZE nonlinear behavior, characterized by a growth exponent $\beta = 2/3$ and a skewness of mean value $\simeq -0.285$ (insert, solid line).

from random initial conditions far from the center of the Rössler attractor, all oscillators quickly reach nearby values, yielding an initially quasi-flat interface.

For system sizes $L \leq 2048$, saturation of the width of the interface could be observed, as well as the linear scaling with system size (Fig. 2a). Regarding the growth of w in a large system, even for the largest size ($L = 2^{18}$) and the longest times ($t \sim 5 \times 10^5$) considered, only the linear regime ($w^2 \sim t^{1/2}$) seems to be observed (Fig. 2b). From these results, we also estimate $D/24\nu \simeq 1.5 \times 10^{-5}$ and $D/\sqrt{2\pi\nu} \simeq 4 \times 10^{-5}$, and thus $\nu \simeq 8 \times 10^{-2}$, $D \simeq 3 \times 10^{-5}$.

To estimate λ , we performed tilt experiments, preparing initial conditions with a prescribed winding number n , and measuring the velocity of the phase interface. Following Eq. (12), only the small q behavior is of interest. However, for too small tilts, the variations of v are not numerically measurable. Consequently, a reliable measure of λ is very difficult. Our results, obtained for moderate tilts, give $\lambda \simeq -2.7$. This is in rough agreement with [2], since, in the stable CGLE context, $\lambda = 2(\beta - \alpha)$ with the parameters estimated at $\alpha \simeq 0.66$ and $\beta \simeq -1.06$.

Gathering these results together, we obtain the following estimates: $g \sim 0.4$, $L_c \sim 350$, and $t_c \sim 2 \times 10^{-4}$. Clearly, there is a contradiction between these estimates and the recorded behavior, which remained in the linear regime well beyond these scales. We will come back to this point in the discussion.

There is one remarkable fact in the above results: the largest widths reached

during our calculations are always very small (at most of the order of 2π). “Roughening” is thus extremely weak in this system, even though the natural extrapolation of our numerical results is that, indeed, phase coherence should be lost in the infinite-size, infinite-time limit. In the next section, we consider the same system but with a modified coupling designed to increase the roughening of the phase interface.

4 Chain of Rössler systems with diffusive-dispersive coupling

For the two-dimensional lattice of [2], the effective CGLE was found to be in a regime where the spatially-homogeneous solution $A = \exp(-i\beta t)$ is linearly stable. One way of bringing the effective CGLE into an intrinsically chaotic regime—and thus, hopefully, to strengthen roughening—is to increase α , so as to be in a phase turbulence regime (which is reached when $1 + \alpha\beta < 0$). For the Rössler lattice, this can be naively achieved by introducing a dispersive-like coupling between the c_1 and c_2 variables, replacing (1) by:

$$\dot{C}_{t+\tau}^i = \mathcal{F}_\tau(C_t^i) + \frac{\varepsilon}{2} \begin{pmatrix} 1 & -\delta & 0 \\ \delta & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \left(\mathcal{F}_\tau(C_t^{i-1}) - 2\mathcal{F}_\tau(C_t^i) + \mathcal{F}_\tau(C_t^{i+1}) \right), \quad (13)$$

where δ is a new parameter controlling the dispersive part of the coupling.

We studied the dynamic scaling of the phase interface with $\delta = 0.4$, and all other parameters as in the previous section. There is a clear increase in the phase fluctuations, though without the appearance of defects, as might be expected from a “phase turbulence-like” behavior. As before, the mean saturated square width scales with L (Fig. 3a), yielding $D/24\nu \simeq 2.85 \times 10^{-5}$. The growth regime of the phase interface of a large system quickly reaches the scaling regime (9) characteristic of the KPZE (Fig. 3b). Even though the roughening remains weak in absolute terms, the $2/3$ exponent indicates that the particular choice of coupling made here achieved its goal. The insert of Fig. 3b shows that the skewness of the interface, a universal ratio of amplitudes, takes the value expected for the one-dimensional KPZE.

The scaling laws for w shown in Fig. 3 do not allow the independent determination of ν and D . From Fig. 3b, using (9), one gets $\lambda D^2/4\nu^2 \simeq 1.2 \times 10^{-7}$. This, together with the value $D/24\nu \simeq 2.85 \times 10^{-5}$ measured from Fig. 3a, actually provides the following estimate: $|\lambda| \simeq 1.0$. There exist several ways of completing the estimation of the KPZE parameters. Here, as we merely wanted to check the consistency of the KPZE picture, we limited ourselves to a rough fit of the early-time ($t < 300$) growth with the linear regime (11) (not shown). This gives $D/\sqrt{2\pi\nu} \simeq 2 \times 10^{-4}$.

Finally, we find $\nu \simeq 0.5$ and $D \simeq 3.5 \times 10^{-4}$, leading to $g \simeq 2.4 \times 10^{-3}$, $L_c \simeq 6 \times 10^4$, and $t_c \simeq 8 \times 10^7$. While the value of L_c is reasonable in view of our numerical results, that of t_c seems too large. One must keep in mind, of course, that these values are only rough estimates, especially for t_c , given their large variation with parameters ν , D , and λ (cf. Eq. (10)). An additional quantitative agreement with the KPZE is provided by the value of the skewness of the interface in the growth regime, which is very close to the “universal” value for the KPZE [7].

5 Discussion

Extrapolating the results of the numerical experiments reported in this work to the infinite-size, infinite-time limit, one may first conclude that phase interfaces of chains of coupled Rössler systems roughen, even if quantitative agreement with the KPZE is debatable. This implies the loss of the phase coherence observed in finite systems. But this roughening is extremely weak, especially in the case of pure diffusive coupling⁵. Using properties of the KPZE, one can only expect an even weaker roughening in two space dimensions. In particular, a very slow logarithmic variation of the saturated width with L during an extremely extended linear regime should be observed (the crossover scales can easily be huge, given their variation with parameters for $d = 2$) [8,9]. It is not surprising, then, that no loss of coherence could be detected within the size/time range investigated in [2].

As mentioned, the validity of the KPZE as the relevant large-scale stochastic description has not been firmly established. While the situation is satisfactory in the case of diffusive-dispersive coupling, there are discrepancies for the purely diffusive case: notably the estimates for L_c and t_c are inconsistent with the fact that the system was observed to remain in the linear growth regime for $L = 2^{18}$ and $t > 10^5$. There are, in our view, two possible reasons for this. First, our estimates of the parameters of the effective KPZE might be inaccurate, leading to estimates for the crossover scales that are orders of magnitude away from their actual values. Indeed, given expressions (10), the values of t_c and L_c can change dramatically even with moderate changes of λ , D , and ν . Moreover, λ is given by the variation of the interface velocity near zero tilt (Eq. (12), a region difficult to probe numerically. Thus, the “true” value of λ could be extremely small, and consequently, L_c and t_c much larger than the estimates found here. Second, and this is probably related to the first point, the “effective noise” could well be very different from (6) [10]. We stress again that the chaotic regime of the Rössler system used here is characterized

⁵ In all the numerical experiments reported here, the width of the phase interface remained smaller than 2π .

by strong amplitude fluctuations (in the (c_1, c_2) plane) and quasi-nil phase fluctuations. Thus, the very strong coherence of the phase interface in the case of diffusive coupling should not be surprising. On the other hand, the cross-coupling term added to introduce dispersion (Sec. 4) does provide a way of obtaining large phase fluctuations directly coupled to the local amplitude chaos of the Rössler system.

Finally, we would like to come back to the status and role of the CGLE in the problem studied here. Even though lattices of coupled Rössler systems do exhibit many of the qualitative features of the CGLE, their equivalence with a CGLE submitted to some noise cannot be a strict one. Specifically, there are no phase soft modes in the Rössler case (the “gauge invariance” of the CGLE is broken). The approximate correspondence between the (c_1, c_2) coordinates of the Rössler system and the complex field A of the CGLE overlooks the role of the c_3 variable. A rough interface must, at every moment, include points where c_3 experiences a sharp peak (Fig.1b). The effect of such localized structures might well be the cause of peculiar properties of the “effective noise” in a Langevin-like description. Since an initially flat interface probably resists the appearance of such structures, one can imagine a particularly strong rigidity of the interface yielding small widths, and, ultimately, very small values of $|\lambda|$.

For the diffusive-dispersive coupling case, on the other hand, the equivalent CGLE is expected to be in a phase turbulent regime⁶, which was shown in [9] to be itself well described by the KPZE at large scales. Any additional perturbations, such as those introduced by the c_3 variable, are not expected to alter significantly this picture, in agreement with our findings.

Even though finer numerical investigations are needed to resolve the difficulties encountered above, our work once more points at the subtleties involved when one tries to build a Langevin description of chaotic extended systems.

Acknowledgements

We thank CESUP-UFRGS (Porto Alegre) and CEA (Saclay) for computing time allowances. LGB wishes to thank CEA-Saclay, where part of this work was performed, for hospitality. HC thanks Cray-Brazil and UFRGS for making his stay in Porto Alegre possible.

⁶ The interface velocity changes little with the dispersive coupling, so that $\beta \simeq 1.06$. Using $|\lambda| \simeq 1$ and assuming, following the stable CGLE, that $\lambda = 2(\beta - \alpha)$ —which is roughly verified for the CGLE even in phase turbulence regimes [9]— we have $1 + \alpha\beta < 0$ and effective CGLE parameters in the phase turbulence region.

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